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*“More geometrico 3 – De l’univers au monde quantique: structures géométriques et topologiques”/“From the Universe to the quantum world: geometric and topological structures”*

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This seminar addressed various topics and different issues related essentially to cosmology and quantum field theories. It was intended to present recent works and discuss new ideas and results from these topics. It has focused on the subject of the geometric and topological structures and invariants which enriched in an remarkable way cosmology and quantum field theories in the last century, say, starting from Einstein’s general relativity until string theory. In the last three decades, new and deep developments in this direction have emerged from cosmology and theoretical physics.

The general aim of the meeting was to examine the role of geometrical and topological concepts and methods in the developments of theoretical physics, especially in cosmology, quantum field theory, string theory, quantum gravity and non-commutative geometry, and then to show the great significance of these concepts and methods for a better understanding of our universe and the physical world at the very small scale. Several speakers stressed the crucial fact that many physical phenomena appear to be related to deep geometrical and topological invariants, and furthermore that they are effect which emerge, in a sense, from the geometric structure of space-time.

The first good example mentioned by various speakers of this new point of view, which however rely upon ideas advocated by Riemann, Clifford and Poincaré, is that of general relativity, which showed that gravity was an effects of the space-time curvature. More precisely, with the general relativity theory, actual (physical) geometry enters the picture of Minkowski space-time (which, mathematically speaking, is a manifold with a Lorentz metric, i.e., a non-degenerate pseudo-Riemannian metric of signature  $+ \dots + -$ ;  $\mathbf{R}^n$  with metric  $(dx^1)^2 + \dots + (dx^{n-1})^2 - (dt)^2$ ) by assuming the world-history of each particle is a geodesic and that the Ricci curvature of the metric reflects the structure of matter and energy present at each point. The Einstein field equations

$$R_{\mu\nu} - 1/2g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}$$

States that mass and pressure warp space-time. These equations relate the metric to matter distribution. Thus, according to the general theory of relativity, the gravitational force has to be reinterpreted as the curvature of space-time in the proximity of a massive object. When the energy is very concentrated, then the deformation of space-time may change sufficiently its topological structure. Let’s stress that general relativity related two fundamental concepts which had, till then, been considered as entirely independent, namely, the concepts of space and time, on the one hand, and the concepts of matter and motion, on the other. Indeed, the geometry of space-time is not given a priori, for, in some sense, matter and its motion

determine it. Conversely, the manner in which the matter behaves depends on the underlying geometric structure of space-time. General relativity theory predicts at least three fundamental phenomena of the physical reality: (i) the gravitational waves; (ii) the black holes; (iii) the expanding of the Universe.

One of the most important ideas of general relativity was that space-time, not space, was the fundamental intrinsic object and that its structure was to be determined by physical phenomena. Einstein's main discoveries were as follow: (i) Spacetime is a pseudo-Riemannian manifold, i.e., its metric  $ds^2$  is not Euclidean but has the signature  $(+,-,-,-)$  at each point. In presence of matter (the gravitational field), general relativity, based on the geometric concepts discovered by Riemann, replaces the flat (pseudo) metric of Poincaré, Einstein (special relativity) and Minkowski,  $ds^2 = dx^2 + dy^2 + dz^2 - dt^2$ , by a curved spacetime metric whose components form the gravitational potential  $g_{\mu\nu}$ ,  $ds^2 = g_{\mu\nu} dx_\mu dx_\nu$ . (ii) Gravitation is just the physical manifestation of the curvature of spacetime (as foreseen by Clifford). (iii) Light travels along geodesics. Another point should be added. The metric of (flat) space-time is not Euclidean but has the form  $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$  at each point. This is what nowadays is called a Lorentzian structure. However, even in the absence of matter, the geometry of space-time could not be asserted to be flat but only Ricci flat, i.e., that its Ricci tensor, which can be calculated from the Riemannian curvature tensor, is 0.

The next essential advance in twenty-century physics has been quantum mechanics. Let's summarize some fundamental idea of this theory. In quantum mechanics and relativistic quantum field theory formulated by P. Jordan, W. Pauli, P. Dirac and E. Wigner (at the subatomic scale), the position and velocity of a particle are non-commuting operators acting on a Hilbert space, and classical notions such as "the trajectory of a particle" do not apply. In the 19<sup>th</sup> and early 20<sup>th</sup> century physics, many aspects of nature were described in terms of fields—the electric and magnetic fields, and the gravitational field—. So since fields interacts with particles, to give an internally coherent account of nature, the quantum concepts must be applied to fields as well as to particles. When this is done, quantities such as the components of the electric field at different points in space-time become non-commutative. When one constructs a Hilbert space in which these operators act, one finds many surprises. The distinction between fields and particles break down, since the Hilbert space of a quantum field is constructed in terms of particle-like excitations. Conventional particles such as electrons, are reinterpreted as arising from the quantization of a field. In the process, one finds the prediction of "antimatter"; for every particle there must be a corresponding antiparticle, with the same mass and opposite electric charge.

The quantum field theories (QFT's) that have proved to be very important in describing elementary particle physics are gauge theories. The classical example of gauge theory is the theory of electromagnetism. The gauge group is the Abelian group  $U(1)$ . If the (physical) potential  $A$  denotes the  $U(1)$  gauge connection, which locally can be regarded, mathematically speaking, as a one-form on space-time, then the curvature or electromagnetic field tensor is the two-form  $F = dA$ , and Maxwell's equations read:  $0 = dF = d^*F$ . Here  $*$  is the Hodge duality operator.

The second main step of the geometrization of physics in the twenty-century has been gauge theory, thanks to which several new deep geometrical and topological structures have emerged. Gauge theory is a quantum field theory obeying to the geometrical principle of local gauge invariance. Gauge theory was introduced by Hermann Weyl in 1918 as an

attempt to unify general relativity with electromagnetism. However the theory of Weyl failed because of lacking of an appropriate quantum physics framework. Gauge idea rebirths with the formulation of non-Abelian Yang-Mills theory in 1954. This new theory stems from the recognition of the structural similarity, from the mathematical viewpoint, of non-Abelian gauge (quantum) fields with general relativity and the understanding that both are connections. This last, defined over a fiber bundle and possessing a curvature, is very deep geometrical concept introduced by Weyl and Cartan, which generalize the concept of parallel transport of Levi-Civita to a new mathematical object: that of a non-point-like space or manifold in which precisely the points are replaced by the fibers.

The very idea of Yang and Mills consists in suggesting a new program of geometrization of physics, this time applied to the physical forces supporting the quantum world. They proposed that the strong nuclear interactions be described by a (quantum) field theory in the same manner than electromagnetism, which is exactly local gauge invariant, as it is general relativity. More precisely, they postulated that the local gauge was the  $SU(2)$  isotopic spin-group or  $SU(2)$  isotopic spin-connection on which the non-Abelian group (a compact Lie group) acts. This idea was “revolutionary” because it changed the very concept of “identity” of what has been ever assumed to be an “elementary particle”. The novel idea that the isotopic spin connection, and therefore the potential  $A_\mu$  (where, in order to relate the phases function  $\lambda(x^i)$  at different points, the familiar gauge transformation for  $A_\mu$  was written in terms of the phase change:  $A_\mu \rightarrow A_\mu - 1/e \partial_\mu \lambda$ ) acts like the  $SU(2)$  symmetry group is the most important result of Yang-Mills theory. The concept of isotopic-spin connection lies at the heart of local gauge theory. It shows explicitly how the gauge symmetry group is built into the dynamics of the interaction between particles and fields. Moreover, some of the important physical characteristics of the field can be deduced directly from the connection (the potential), which can be viewed as a linear combination of the generators of the  $SU(2)$  group. We can, in fact, associate this formal operation with real physical processes.

Let’s add few specifications on the mathematical structure of gauge theory. Yang-Mills or non-Abelian gauge theory can, at the classical level, be described similarly to the “classical” Abelian gauge theory, with  $U(1)$  (see above) replaced by a more general compact gauge group  $G$ . The definition of curvature must be modified to  $F = dA + A \wedge A$ , and Maxwell’s equations are modified to the Yang-Mills equations:  $0 = d_A F = d_A * F$ , where  $d_A$  is the gauge-covariant extension of the exterior algebra derivative. These equations can be derived from the Yang-Mills Lagrangian.

$$\mathcal{L} = 1/4g^2 \int \text{Tr} F \wedge *F,$$

where  $\text{Tr}$  denotes an invariant quadratic form on the Lie algebra of  $G$ . The Yang-Mills equations are non-linear, so, in contrast to the Maxwell equations, but like the Einstein equations for the gravitational field, they are not explicitly solvable in general. But they have certain properties in common with the Maxwell equations and, in particular, they describe at the classical level massless waves that travel at the speed of light.

The first (classical) Yang-Mills theory corresponds to the quantum version of Maxwell theory—known as Quantum Electrodynamics—, which gave a very accurate account of the quantum behaviour of electromagnetic fields and forces. The non-Abelian gauge theory for describing the other forces in nature, notably the weak force (responsible among other things for certain forms of radioactivity) and the strong or nuclear force (responsible among other

things for the binding of protons and neutrons into nuclei). For the weak force, we have now the Weinberg-Salam-Glashow electroweak theory with gauge group:  $H = SU(2) \times U(1)$ .

The masslessness of classical Yang-Mills waves was avoided by elaborating the theory with an additional “Higgs field”. This is a scalar field, transforming in a two-dimensional representation of  $H$ , whose non-zero and approximately constant value in the vacuum state reduces the structure group from  $H$  to  $U(1)$  sub-group (diagonally embedded in  $SU(2) \times U(1)$ ). This theory describe both the electromagnetic and weak forces, in a more or less unified way; because of the reduction of the structure group to  $U(1)$ , the long range fields are those of electromagnetism only, in accord with what we see in nature.

To sum up what we said about gauge theory, let’s stress that Yang and Mills showed for the first time that local gauge symmetry was a powerful fundamental principle that provided new insights into the newly discovered “internal” quantum numbers like isotopic spin. In their theory, isotopic spin was not just a label for the charge states of particles, but it was crucially involved in determining the fundamental forms of the interaction between these particles. The most important philosophical point is that in the gauge theories of quantum fields symmetries of nature determine the properties of forces; therefore, it is allowed to say that mathematical groups and invariants are at the origin of the dynamics of physical forces.

Let’s add that in the search for a non-linear generalization of Maxwell’s equations to explain elementary particles, there are various symmetry properties one would require. These are: (i) External (spatio-temporal) symmetries invariant under the Lorentz and Poincaré’s groups and under the conformal group if one is taking the rest-mass to be zero; (ii) Internal (physical) symmetries invariant under the non-Abelian groups like  $SU(2)$  or  $SU(3)$  to account for the known feature of weak and strong interactions, respectively; (iii) Covariance or its supersymmetric coupling by working on a complex topological space-time.

The next fundamental step in the geometrization of physics has been realized by string theory, a quantum field theory that tries to unify in a coherent picture general relativity and quantum mechanics at a deeper level than that of the Standard Model of particle physics. String theory entails beautiful geometrical and topological new structures, more rich and powerful with respect to those developed before by the other quantum field theories. It is yet theoretically incomplete and hitherto physically untested.

It is worth of recalling that originally string program go back, in a sense, to the ideas putted forward by the German mathematician B. Riemann about hundred-fifth years early. According to which, one ca make two fundamental assumptions. (i) First, on a given  $n$ -dimensional manifolds there are many possible metric structures (i.e., many different functions for measuring the distance between any pair of infinitesimally near points), so that the problem of which structure is the one appropriate for physical space required empirical methods for its solution; in other words, Riemann stated explicitly (1854) that the question of the geometry of physical space does not make sense independently of physical phenomena; and (ii) Space does not exist independently of phenomena and its structure depends on the extent to which we can observe and predicts what happens in the physical world. From the previous follows, say, a corollary even more insightful: in its infinitely small parts (nowadays we would say at the quantum level) space may not be accurately described even by the geometrical notions of Riemannian geometry.

This last idea, which is hinted at in Riemann’s statement (ii), may dormant until the

search for a unified field theory at the quantum level forced the physicists to reconsider the structure of space-time at extremely small distances. One of the ideas to which their efforts led them was that the geometry of spacetime was supersymmetric with the usual coordinates supplemented by several anticommuting (fermionic) ones. This is a model that reflects the highly fuzzy structure of spacetime in small regions (at the quantum scale  $10^{-33}$  cm) where one can pass back and forth between bosonic and fermionic particles. Modern string theory (i.e., superstring theory) takes Riemann's vision even further, and replaces the points of spacetime by strings, thereby making the geometry even more non-commutative. As we shall report below, Gianni Landi devoted its talk exactly to this truly important topic.

Let's address briefly some conceptual aspects and issues of string theory. Superstring theory relies on the two ideas of supersymmetry and spacetime structure of eleven dimensions. Supersymmetry requires that for each known particle having integer spin 0, 1, 2, and so on, measured in quantum units—there is a particle with the same mass but half-integer spin ( $1/2$ ,  $3/2$ ,  $5/2$  and so on), and vice-versa. Supersymmetry transforms the coordinate of space and time such that the laws of physics are the same for all observers. Einstein's general theory of relativity derives from this condition, and so supersymmetry implies gravity. In fact, supersymmetry predicts "supergravity", in which a particle with a spin of 2—the graviton—transmits gravitational interactions and has as a partner a gravitino, with spin of  $3/2$ .

Superstring is based on the fundamental notion of T-duality, which relates two kinds of particles that arise when a string loops around a compact (spatial) dimension. One kind, call them "vibrating particles", is analogous to those predicted by Kaluza and Klein and comes from vibrations of the loop of the string. Such particles are energetic if the circle is small. In addition, the string can wind many times around the circle, its energy becomes higher the more times it wraps around and the larger the circle. Moreover, each energy level represents a new particle—call them "winding particle". T-duality states that the winding particles for a circle of radius  $R$  are the same as the "vibrating particles" for a circle of radius  $1/R$ , and vice-versa. So, to a physicist, the two sets of particles are indistinguishable: a fat compact dimension may yield apparently the same particles as thin one.

String theory, if correct, entails a radical change in our concepts of spacetime. That is what one would expect of a theory that reconciles general relativity with quantum mechanics. The answer involves duality again. A vibrating string is described by an auxiliary two-dimensional field theory, whose Lagrangian is roughly

$$\mathcal{L} = \frac{1}{2} \int dt d\sigma (\partial X/\partial \tau)^2 + (\partial X/\partial \sigma)^2.$$

Here,  $X(\tau, \sigma)$  is the position of the string at proper time  $\tau$ , at a coordinate  $\sigma$  along the string. In string theory, the auxiliary two-dimensional field theory plays a more fundamental role than spacetime, and spacetime exists only to the extent that it can be reconstructed from the two-dimensional theory. In other words, duality symmetries of the two-dimensional field theory put a basic restriction on the validity of the classical notion of spacetime.

All the attempts mentioned, which are aimed at solving one of the central problems in twentieth-century physics, i.e.: how to combine gravity and the other forces into an unitary theoretical explanation of the physical world, essentially depend on the possibility of building a new geometrical framework conceptually richer than Riemannian geometry. In fact, as we saw, it plays a fundamental role in non-Abelian gauge theories and in superstring

theory, thanks to which a great variety of new mathematical structure has emerged. A very interesting hypothesis is that the global topological properties of the manifold's model of spacetime play a major role in quantum field theory and that, consequently, several physical quantum effects arise from the non-local metrical and topological structures of these manifold. Thus the unification of general and quantum theory require some fundamental breakthrough in our understanding of the relationship between spacetime and quantum processes. In particular the superstring theory, but also, in a different manner, loop quantum gravity, lead to the guess that the usual structure of spacetime at the quantum scale must be dropped out from physical thought. Non-Abelian gauge theories satisfy the basic physical requirements pertaining to the symmetries of particle physics because they are geometric in character. They profoundly elucidate the fundamental role played by bundles, connections and curvature in explaining the essential laws of nature. Kaluza-Klein theories and more remarkably superstring theory showed that spacetime symmetries and internal (quantum) symmetries might be unified through the introduction of new structures of space with a different topology. This essentially means that "hidden" symmetries of fundamental physics can be related to the phenomenon of topological change of certain class of (presumably) non-smooth manifolds. This entails a number of extremely important mathematical and physical consequences, which partly have been discussed during the meeting.

Several talks addressed some of the recent most fundamental developments in mathematical and theoretical physics, and in particular, the fact that these developments point forwards the search for a new scheme of spacetime structure at the quantum scale. Quantum mechanics culminated in the "standard model" of particle interactions, which is a quantum field theory. The fundamental ingredients of nature that appear in the underlying equations are fields: the familiar electromagnetic field, and some twenty or so other fields. The so-called elementary particles, like photons and quarks and electrons, are "quanta" of the fields-bundles of the fields energy and momentum. The properties of these fields and their interactions are largely dictated by principles of symmetry, including Einstein's special principle of relativity, together with a principle of "renormalizability", which dictates that the fields can only interact with each other in certain specially ways. The standard model has passed every test that can be imposed with existing experimental facilities. However many unsolved problems and open questions remain. We do not know why it obeys certain symmetries and not others, or why it contains six types of quarks, and not more or less. Finally, gravitation cannot be brought into the quantum field theoretic framework of the standard model, because gravitational interactions do not satisfy the principles of renormalizability that governs the other interactions. This constitutes at present one the most fundamental and challenging issues of researches in theoretical physics and mathematics. Both topological quantum field theories and non-commutative geometry dedicate much effort to find out a solution to the very hard and key problem of the renormalization of the standard model. In its talk, Gianni Landi mentioned this problem and he sketched the Conne's non-commutative approach to getting an answer.

The not-yet-achieved incorporation of the fundamental ideas of a dynamical space-time geometry into a quantum theory of matter is one of the central open problems of contemporary physics, whose solution may well require another radical change in the physicist's conception of nature and space-time. We think that a real understanding of the cosmological questions and of the nature of elementary particles can ever been achieved

without a simultaneous deeper understanding of the nature of space-time itself. It is well-known that quantum mechanics taught us that the classical notions of the position and velocity of a particle were only approximations of the truth. Notably, it is not clear whether the Riemannian geometry—even in a revised and generalized form—is adequate for the description of the small scale structure of space-time. The Planck length  $l_p = (Gh/c^3)^{1/2} \sim 10^{-33}$  cm is considered as a natural lower limit for the precision at which coordinates of an event in space-time make sense. Nevertheless, not only does quantum mechanics have some striking geometrical characters, but its description of the world also reveals a wealth of deep underlying mysteries—even bordering on paradox—which cannot arise merely from an inadequate human understanding of the implications of the theory’s mathematical formalism. Instead, at some level, there must be a deviation from purely unitary evolution, so that state-vector reduction can become a real phenomenon. Moreover, because of the (mysterious) non-local nature of quantum entanglement, whatever the nature of this revolution might be, the final theory that will emerge must have a fundamentally non-local character. In effect, according to certain mathematical-physical theories, such as topological quantum field theories and especially superstring theory, the local information of the space-time fields and of the other fields is stored in global (topological) structures of space-time.

This is also truth for non-commutative geometry, where the quantum field equations are calculated for the full set of internal space metric fluctuations allowed by the non-commutative geometry axioms in the spectral triple formulation of the standard model. These calculations have been given both from the perspective of the spectral triple and from the perspective of Fredholm module<sup>1</sup>. It has been showed that studying these Fredholm modules using algebraic  $K$  theory and  $K$  homology leads to a suggested non-commutative version of Morse theory—a well-known tool for studying the topology of manifolds—which is applied to the finite spectral action. According to the spectral action principle, which has been introduced ten years ago by Connes and Chamseddine, the standard model of particle physics is formulated with a product (whose image is called the total space) of two spectral triples—one that represents the Euclidean space-time manifold and the other the zero-dimensional internal space of particle charges. The space-time coordinate functions remain commutative but the internal space is a noncommutative “manifold”. The spectral action principle is an important step towards the unification of gravity with particle physics; the Einstein-Hilbert action plus Weinberg-Glashow-Salam theory all result from a calculation of the eigenvalues of the Dirac operator on the total space and since the Dirac operator encodes the metric, the spectral action principle is a purely geometrical theory.

Formally, a spectral triple  $(\mathcal{A}, \mathcal{H}, \mathcal{D})$  provides the analog of a Riemannian spin manifold to noncommutative geometry. It consists of an involutive, non-necessarily commutative algebra  $\mathcal{A}$ , a Hilbert space  $\mathcal{H}$ : a finitely generated projective module on which the algebra is

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<sup>1</sup> Recall that the Fredholm module is the “premetric” structure that is used to define the non-commutative calculus. A spectral triple can be thought of as an unbounded (unless the Hilbert space is finite dimensional) Fredholm module with Dirac operator providing a notion of distance. More formally, if  $P$  is an elliptic  $m^{\text{th}}$  order differential operator on a compact manifold then  $P$  has a finite dimensional kernel, but  $P$  elliptic implies  $P^*$  elliptic and so  $P^*$  also has a finite dimensional kernel. Thus we can say that  $P$  has the property of possessing a finite dimensional kernel and cokernel. When  $P$  is viewed as an operator between Sobolev spaces this remains true, but bounded linear operators between Hilbert spaces with this property are called Fredholm operators. Thus elliptic differential operators  $P$  are also Fredholm when realized on Sobolev spaces.

represented, and a Dirac operator  $\mathcal{D}$  that gives a notion of distance, and from which is built a differential algebra. A very important technical point is that the geometry of any closed (even dimensional) Riemannian spin manifold can be fully described by a (real and even) spectral triple and a noncommutative geometry is essentially the same structure but with the generalization that the algebra of coordinates are allowed to be non-commuting. For the standard model the internal Hilbert space is  $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^C \oplus \mathcal{H}_R^C$ , where

$$\mathcal{H}_L = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N),$$

$$\mathcal{H}_R = ((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N),$$

and whose basis is labeled by the elementary fermions and their antiparticles. The symbol  $c$  is used to indicate the section represented by the antiparticles. The even triple has the  $\mathbb{Z}/2$ -grading operator  $\chi$ , the chirality (eigenvalues  $+1$  or  $-1$ ). In either case of  $\mathcal{H}_L$  and  $\mathcal{H}_R$ , the first direct summand is the quarks and the second the leptons.  $N$  stands for the numbers of generations. For example, the left-handed up and down quarks form an isospin doublet and their right-handed counterparts are singlets and there are three colors for quarks and none for leptons. The charges on the particles are identified by the faithful representation of the algebra on the Hilbert space. In the definition of  $\mathcal{H}$  above we see a second  $\mathbb{Z}/2$ -grading that splits the Hilbert space into two orthogonal subspaces for particles and antiparticles:  $\mathcal{H}^+ \oplus \mathcal{H}^-$  or  $\mathcal{H} \oplus \mathcal{H}^C$ . This is called  $S^0$  reality and is not an axiom but applies to the standard model as it excludes Majorana masses. The  $S^0$  reality grading operator  $\varepsilon$  satisfies:  $[\mathcal{D}, \varepsilon] = 0$ ,  $[J, \varepsilon]_+ = 0$ ,  $\varepsilon^* = \varepsilon$ ,  $\varepsilon^2 = 1$ .

Some philosophical aspects related to the central issues of the above mathematical-physical approaches to the uncovering of what we call the “physical reality”, have been discussed, from different approaches and perspectives, by several speakers, in particular by Gabriel Catren, Frédéric Patras, Alexei Grinbaum, Marc Lachièze-Rey and Sisir Roy. Let’s just address the important point concerning the differences between the “ontology” (here this term stand for the nature and the kind of properties ascribed to the most fundamental physical entities from which a specific theory is built up and also to the mathematical objects by means of which one construct a definite space-time theory or model) of classical physics and quantum physics. One may affirm that Newtonian physics had a clear ontology: the world consisted of massive particles situated in Euclidean space. In that sense, the nature of space played a fundamental role. In the mathematical developments of Newtonian mechanics, however, the role of space is not clear. There is not much difference between the description of two particles moving in  $\mathbf{R}^3$  and that of a single particle moving in  $\mathbf{R}^6$ , nor between that of a pivoted rigid body and that of a point moving on the group-manifold  $SO_3$ . In quantum mechanics the idea of space is even more elusive, for there seems to be no ontology, and, whatever wave-functions are, they are certainly not functions defined in space. Still, for about seventy years we have known that elementary particles must be described not by quantum mechanics but by quantum field theory, and in the field theory the role of space is quite different. Although it is an important fact that quantum field theory cannot be reconciled with general relativity, one could emphasize that the two theories have a virtual feature in common, for in both of them the points of space play a central and objective dynamical role. In quantum field theory two electrons are not described by a wave-function on  $\mathbf{R}^6$ ; instead they constitute a state of a field in  $\mathbf{R}^3$  which is excited in the

neighborhood of two points. The points of space *index* the observables in the theory. The mathematics of quantum field theory is an attempt to describe the nature of space, but it proposes to look at space in a completely different way. Like quantum field theory, Penrose's twistor theory is a radical attempt to get rid of space as a primary concept. The Connes's programme of non-commutative geometry amounts to a huge generalization of the classical notion of a manifold. Finally, string theory proposed a scheme for making space as an approximation to some more general kind of structure. One striking difference (maybe the essential one) between general relativity and quantum mechanics lie in the fact that, whereas in general relativity it seems impossible to separate the postulate of (continuous) space-time localization of events and the theory of gravitation from the (inner) geometric structure of space-time, on the other hand, it is precisely these postulate of the indistinguishability of the physical fields from the space-time geometry that got lost in quantum mechanics. It is particularly contradicted by the Bohr principle of complementarity and the Heisenberg uncertainty relations, which states the impossibility of knowing simultaneously the exact position and velocity of particles (electrons). These relations are indeed based on a model in which the electron jumps quickly from one orbit to another, radiating all energy thus liberated in the form of a global package, a *quantum* of light.

Many attempts have been made, starting from the sixties, to understand what kind of geometry and topology and therefore what kind of space-time model could be truly appropriate to describe the behaviour of physical space both at the very large and quantum levels. Among them, the most attractive and promising ones seem to be string theory, non-commutative geometry and loop quantum gravity. The nature of quantum geometry is the central issue of non-perturbative quantum gravity. Is the familiar continuum picture then only an approximation? If so, what are the 'atoms' of geometry? What are its fundamental excitations? Is there a discrete underlying structure? If so, how does the continuum picture arise from this fundamental discreteness? By a quantized geometry, it is meant (according notably to Ashtekar, Rovelli, Smolin) that there exist physical quantities which can take on continuous values classically but are such that the corresponding quantum operators have a discrete spectrum. In the resulting quantum geometry, Riemannian geometry can then emerge only as an approximation on a large scale. The very interesting and enjoyable talk by Mauro Carfora was essentially devoted to discuss this topic, from the main perspective of topological quantum field theory and superstring theory.

The most attractive feature of non-commutative geometry is that it develops a new notion of geometric space where points do not play the central role, thus giving much more freedom for describing the subatomic-scale nature of spacetime. The theory proposed a framework which is sufficiently general to treat discrete space, Riemannian manifolds, configurations spaces of quantum field theory, and the duals of discrete groups which are not necessarily commutative. The development of a non-commutative geometry has been a very important import on the recent attempts to unify (mathematically) quantum field theory with gravitation. In addition, its physical implications have found lately a confirmation in that it predicted a physical model for coupling gravity with matter.

The other fundamental change in our conception of spacetime and physics comes from superstring theory. Indeed, recent developments in theoretical physics suggest that a new kind of quantum geometry may enter physics, and that spacetime itself may be reinterpreted as an approximate, derived concept that one can extract from a two-dimensional field

theory. Intuitively, strings are viewed as one-dimensional objects whose modes of vibration represent the elementary particles. In addition, in string theory the one-dimensional trajectory (world-line) of a particle in space-time is replaced by a two-dimensional orbit (world-tube) of the string. The main conceptual point of the string programme is that it entails some revolutionary ideas about our conception of space and space-time. Indeed, space is not more thought as formed up of points-like elements and therefore the particles not either. Also space is endowed with a point-less structure. Instead of point-like elements, the space seems to be filled out of other kinds of geometrical objects, more rich and complex like knots of many types, Riemannian surfaces, topological (unconventional) objects, and so on. The most interesting point is that space must be considered as a dynamical thing, which may change with respect to its metrical and topological properties. The main physical aspect of string theory is that all particles which we previously thought of as elementary, that is, as little points without any structure in them, turn out in fact not to be points at all but basically little loops of string which move through space, oscillating around it. We have thus that the different physical properties of matter are linked to the different structural patterns of these strings and loops in space. For example, the electric charge might be seen as a quality of the motion of the string rather than something which is just added on to a particle as fundamental object.

We now outline some new ideas relating to the structure of space-time in the most recent physical theories, to start with general relativity. (i) The geometric structure of space-time gives rise to the dynamics of this same space-time, and in particular of the gravitational field. (ii) Even the other (fermionic and bosonic) fields describing matter and its electroweak and strong interactions seems to emerge as dynamical effects from the topological (global) structure of space-time. Conversely, the space-time itself must be henceforth thought of, in some sense, as a derived (changing) object whose metric and topological structures may be subject, to some extent, to the quantum fluctuations of these same fields. For example, one of the prediction of T-duality in string theory is that geometry and topology are *ambiguous* at the string length  $l_S = \sqrt{\alpha'}$ . Further more, space is ambiguous at the Planck length  $l_P \ll l_S$ . Another more complicated and richer example of T-duality is the mirror symmetry and topology change in Calabi-Yau spaces. Joseph Kouneiher's talk was mostly devoted to review the role played by different types of dualities in the recent developments of theoretical physics. One conclusion is, thus, that spacetime is likely to be an emergent, approximate, classical concept. The challenge is to have emergent spacetime, while preserving some locality (macroscopic locality, causality, analyticity, etc.). (iii) The recent developments of theoretical physics enables us to think that the discrete and continuous character of the laws of physics are but special cases according with each other in the framework of a new unitary mathematical-physical theory. With the theory of supergravity, and still more with string theory, we get a consistent theoretical framework which is finite and which simultaneously incorporate both quantum gravity and chiral supersymmetric gauge theories in a natural fashion. Supergravity generalizes a gauge theory proposed by H. Weyl in 1923 in order to unify the Einstein's theory of gravitation with the electromagnetic theory, and another by Kaluza and Klein in the 1920s, in which they suggested to further unify the concepts of internal and space-time symmetries by reducing the former to the latter through the introduction of some extra dimension of space, more precisely, a fifth (space-like) dimension, which has the topology of a circle. (iv) The

physical (dynamical) and space-time symmetries dictate, at different extents, the various forces of nature and the interactions between particles. This is a very general principle and it is the crucial idea at the heart of quantum field theories. In fact, all physical phenomena seem to be founded upon such principle. However, at a deeper level, one is increasingly led to believe that, beside symmetries (including, space-time, physical, broken symmetries, and maybe other “hidden” symmetries), topological structures and invariants might have an even more important role in determining physical phenomena at the very large and extremely small scales.

To conclude this report, let’s stress one again the main topic discussed in the meeting. For the sake of simplicity, we group and summarize them as follows.

– *Topological and geometrical structures of the cosmological models which describe the form of the universe* (J.-P. Luminet and J. Weeks addressed this topic). They presented and discussed, from different viewpoints, the topological model of overall form of the universe based on the *Poincaré Dodecahedron Space* (PDS) As wonderfully showed by J. Weeks, this space is positively curved and topologically finite; it is a multi-connected variant of the simply-connected hypersphere  $S^3$ , with a volume 120 times smaller for the same curvature radius. Weeks also showed how to construct the Poincaré space by help of our mathematical imagination, and further illustrated the most remarkable geometrical and topological properties of it. From the physical point of view, the PDS model provide an explanation for a surprising detail observed in the Cosmic Microwave Background (CMB) recently mapped by the NASA satellite WMAP. According to a study by Luminet and its collaborators published in 2003 in *Nature*, an intriguing discrepancy in the temperature fluctuations in the afterglow of the big bang can be explained by a very specific global shape of space (a “topology”). The universe could be wrapped around, a little bit like a “soccer ball”, the volume of which would represent only 80% of the observable universe. If confirmed, this would be a major discovery about the nature of the universe. Luminet explained that, since some power spectrum anomalies are one of the possible signatures of a finite and multi-connected universe, there is still a continued interest in the Poincaré dodecahedral space and related finite universe models. And even if the particular dodecahedral space is eventually ruled out by future experiments, all of the other models of well-proportioned spaces will not be eliminated as such. In addition, numerical simulations show that, even if the size of a multiply-connected space is larger than that of the observable universe, we could all the same discover an imprint in the fossil radiation, even while no pair of circles, much less ghost galaxy images, would remain. The topology of the universe could therefore provide information on what happens outside of the cosmological horizon! But this is search for the next decade. Luminet stressed that maybe the most fundamental issue remains to link the present-day topology of space to a quantum origin, since classical general relativity does not allow for topological changes during the course of cosmic evolution. Theories of quantum gravity could allow addressing the problem of a quantum origin of space topology. For instance, in the approach of quantum cosmology, some simplified solutions of Wheeler-de Witt equations show that the sum over all topologies involved in the calculation of the wavefunction of the universe is dominated by spaces with small volumes and multiconnected topologies. In the approach of brane worlds, the extra-dimensions are often assumed to form a compact Calabi-Yau manifold; in such a case, it would be strange that only the ordinary dimensions of our 3-brane would not be compact like the extra ones. These are only heuristic indications on the way unified theories of gravity and quantum mechanics could “favor” topologically multiconnected spaces. Whatsoever the fact that

some particular multiconnected space models, such as PDS, may be refuted by future astronomical data, the question of cosmic topology will stay as a major question about the ultimate structure of our universe.

– *The geometry of space-time in general relativity, dark matter and the new cosmological paradigms* (J.-M. Alimi and M. Francaviglia). With general relativity Einstein merged geometry and physics in a new deep manner, paving the way to a new vision in physics where curvature plays a dominant role and linearity represents the exception. Einstein's ideas pervaded most of the physics of the 20th century. General relativity revealed itself as a founding discipline for cosmology, leading to great discoveries about the properties of the Universe. Nevertheless, this theory is nowadays suffering several drawbacks, due to a deeper understanding of physical phenomena, the advent of “gauge theories”, the “string” paradigms, the lack of a coherent and universally accepted quantization scheme for gravity and, finally, because of astronomical observations that seem to indicate a lack of matter up to the 95% of the amount of energy needed to explain the behaviour of the Universe if Einstein's equations are assumed to hold (at least at large scales). At the light of these new discoveries and problems we face a challenging question. M. Francaviglia claimed that Einstein today would renounce to privilege the metric structure as the founding structure of gravitation, reverting to something that was in fact discovered after the birth of general relativity: namely, the notion of “linear connection”. He further explained this important point in more details, both on historical and on physical grounds. J.-M. Alimi made a comprehensive presentation of the cosmological models proposed since the discovering of general relativity to account for the problem of the nature and the meaning of the cosmological constant that appeared for the first time in the Einstein's field equations. Today, evidence is accumulating that most of the energy density in the universe is in an unknown form dubbed dark energy.

The picture proposed recently by string theory explains how all the different stable vacua of the “string landscape” come into existence at various locations in the universe, thus forming innumerable sub-universes. This result might contribute to solve one of the most important and long-standing problems in theoretical physics—one related to the vacuum energy. To Einstein, what we now think of as vacuum energy was an arbitrary mathematical term—a “cosmological constant”—that could be added to his equation of general relativity to make it consistent with his conviction that the universe was static. To obtain a static universe, he proposed that this constant takes a positive value, but he abandoned the idea after observations proved the universe to be expanding. With the advent of quantum field theory, empty space—the vacuum—became a busy place, full of virtual particles and fields popping in and out of existence, and each particle and field carries some positive or negative energy. Concerning the nonzero value of the vacuum energy, the general idea of string theory is that the complicated geometries of hidden dimensions might produce a spectrum for vacuum energy that includes values in the experimental window. If the landscape picture is right, a nonzero vacuum energy should be observed, most likely not much smaller than  $10^{-118} \Lambda_p$  (the value of  $\Lambda_p$  is about  $10^{94}$  grams per cubic centimeter, or one Planck mass per cubic Planck length).

As we just said, Einstein introduced the cosmological constant to his (field) equations of general relativity because he believed the universe was static. Faced with evidence that the

universe was actually expanding, however, he decided to remove it, later referring to the cosmological constant as the biggest blunder of his life. Recent observations suggest that the expansion of the universe is accelerating, which favours a small, but non-zero, positive cosmological constant with a value of  $10^{-120}$  in Planck units. But perhaps Einstein's most serious mistake regarding the cosmological constant,  $\Lambda$ , was to actually believe that he had the right to decide or not it should be included in his equations in the first place. If  $\Lambda$  is very small or zero, its value has to be explained, and the explanation will very likely entail the vacuum's energy and fluctuations issue as a key feature. This is one of the greatest puzzles of modern theoretical physics. Without a cosmological constant, the most symmetric solution of Einstein's equations in the vacuum is the flat, four-dimensional Minkowski spacetime of special relativity. In 1917 the Dutch astronomer Willem de Sitter found the analogous solution if the cosmological constant is non-zero. If  $\Lambda$  is positive, the solution is called "de Sitter space" and  $\Lambda$  is the vacuum energy that curves spacetime; a negative value of  $\Lambda$  corresponds to what is called anti-de Sitter space. (Anti-de Sitter space is a maximally symmetric Lorentzian manifold with constant negative scalar curvature; it is the Lorentzian analog of  $n$ -dimensional hyperbolic space, just as Minkowski space and de Sitter are the analogs of Euclidean and elliptical spaces respectively). Einstein immediately rejected the de Sitter solution because it went against his intuition—it implied that spacetime can be curved in the absence of matter—although he ended up accepting the idea after some debate. In de Sitter space the universe expands exponentially, which is the basis of the inflationary model of the universe. De Sitter space play also an important role in string theory.

J.-M. Alimi explained why regionally relativistic cosmologies are a way to tackle the so-called coincidence problem of standard Friedmannian cosmologies: why is the expansion accelerating approximately at the same time when the Universe becomes structures, that is when the density contrast in the matter field is no longer small on a wide range of scales? The main issue is to try and link the dynamics of the Universe on large scales to its structuration on smaller scales. It consists in building, from a fully non-homogeneous Universe, cosmological models that are homogeneous, thanks to a spatial averaging procedure. It results in equations for a volume scale factor that not only include an averaged matter source term, but also additional terms that arise from the coarse-grained inhomogeneities. These additional terms are commonly named backreaction. He then showed that there is a correspondence between regionally averaged cosmologies and Friedmannian scalar field cosmologies, the scalar field being interpreted in this context as a mean field description of the inhomogeneous Universe, that can play the role of a field responsible for the dark energy phenomenon.

– *Properties and perspectives of topological quantum field theories; geometries of the quantum world. Geometric invariants of low-dimensional manifolds and the connection between them and the most crucial theoretical physics open questions; a particular attention was devoted to the relationship between knot and link theory invariants and Chern-Simons theory* (M. Carfora, L. Boi, A. Marzuoli, S. Roy). Topological quantum field (TQFT) emerged in the eighties as a new relation between mathematics and physics. The relation connected some of the most advanced ideas in the two fields. The nineties have been characterized by its development, originating unexpected results in topology and testing some of the most fundamental ideas in quantum field theory and string theory. The first TQFT was formulated by Witten in 1988. He constructed the theory now known as Donaldson-Witten theory, which constitutes a quantum field theory representation of the

Donaldson invariants of four-manifolds (1983-84). His work was strongly influenced by M. Atiyah. In 1988 Witten formulated also another two TQFTs which have been widely studied during the two decades: topological sigma models in two dimensions and Chern-Simons gauge theory in three dimensions. These theories are related, respectively, to Gromov invariants, and to knot and link invariants as the Jones polynomial and its generalizations. TQFT has provided an entirely new approach to study topological invariants. Being a quantum field theory, TQFT can be analyzed from different point of view. The richness inherent to quantum field theory can be exploited to obtain different perspectives on the topological invariants involved in TQFT. This line of thought has shown to be very fruitful in the last then years and new topological invariants as well new relations between them have been obtained.

TQFT have been studied from both, perturbative and non-perturbative points of view. In the case of Chern-Simons gauge theory, non-perturbative methods have been applied to obtain properties of knot and link invariants, as well as general procedures for their computation. Perturbative methods have also been studied for this theory providing integral representations for Vassiliev invariants. In Donaldson-Witten theory perturbative methods have proved its relation to Donaldson invariants. Non-perturbative methods have been applied after the work by Seiberg and Witten on  $N = 2$  supersymmetric Yang-Mills theory. The outcome of this application is a totally unexpected relation between Donaldson invariants and a new set of topological invariants called Seiberg-Witten invariants.

Donaldson-Witten theory is a TQFT of cohomological type. TQFTs of this type can be formulated in a variety of frameworks. The most geometric one corresponds to the Mathai-Quillen formalism. In this formalism a TQFT is constructed out of a moduli problem. Topological invariant are then defined as integrals of a certain Euler class (or wedge products of the Euler class with other forms) over the resulting moduli space. A different framework is the one based on the twisting of  $N = 2$  supersymmetry. In this case, information on the physical theory can be used in the TQFT. Indeed, it has been in this framework where Seiberg-Witten invariants have shown up. After Seiberg and Witten worked out the low energy effective action of  $N = 2$  supersymmetric Yang-Mills theory it became clear that a twisted version of this effective action could lead to topological invariants related to Donaldson invariants. The twisted action revealed a new moduli space, the moduli space of Abelian monopoles. Its geometric structure has been derived in the context of the Mathai-Quillen formalism. Invariants associated to this moduli space should be related to Donaldson invariants. This turned out to be the case. The relevant invariants for the case of  $SU(2)$  as gauge group are the Seiberg-Witten invariants.

Donaldson-Witten theory has been generalized after studying its coupling to topological matter fields. The resulting theory can be regarded as a twisted form of  $N = 2$  supersymmetric Yang-Mills theory coupled to hypermultiplets, or, in the context of the Mathai-Quillen formalism, as the TQFT associated to the moduli space of non-Abelian monopoles. Perturbative and non-perturbative methods have been applied to this theory for the case of  $SU(2)$  as gauge group and one hypermultiplet of matter in the fundamental representation. In this case, again, it turns out that the generalized Donaldson invariants can be written in terms of Seiberg-Witten invariants. One would expect that in general the invariants associated to non-Abelian monopoles could be expressed in terms of some other simpler invariants, being Seiberg-Witten invariants just the first subset of the full set of

invariants.

The present situation in three and four dimensions relative to Chern-Simons gauge theory and Donaldson-Witten theory, respectively, can be described as follows. These theories share some common features. Their topological invariants are labeled with group-theoretical data: Wilson lines for different representations and gauge groups (Jones polynomials and its generalizations), and non-Abelian monopoles for different representations and gauge groups (generalized Donaldson polynomials); these invariants can be written in terms of topological invariants which are independent of the group and representation chosen: Vassiliev invariants and Seiberg-Witten invariants. This structure leads to the idea of universality classes of topological invariants. In this respect Vassiliev invariants constitute a class in the sense that all Chern-Simons or quantum group knot invariants for semi-simple groups can be expressed in terms of them. Similarly, Seiberg-Witten invariants constitute another class since generalized Donaldson invariants associated to several moduli spaces can be written in terms of them. This certainly holds for the two cases described above but presumably it holds for other groups. It is very likely that Seiberg-Witten invariants are the first set of a series of invariants, each defining a universality class.

– *Quantum field theory and the geometrical structures of the universe; in particular de Sitter and anti-de Sitter spaces* (U. Moschella, F. Patras, M. Lachièze-Rey).

Quantum field theory (QFT) is a theoretical framework for constructing quantum mechanical models of systems classically described by fields (having an infinite number of degrees of freedom) or of many-body systems (having a finite number of degrees of freedom). Most theories in modern particle physics, including the Standard Model of elementary particles and their interactions, are formulated as relativistic quantum field theories.

The quantum field theories (QFT's) that have proved to be most important in describing elementary particle physics are gauge theories. The classical example of a gauge theory is the theory of electromagnetism. The gauge theory is the Abelian group  $U(1)$ . If  $A$  denotes the  $U(1)$  gauge connection, which locally can be regarded as a one-form on spacetime, then the curvature or electromagnetic field tensor is the two-form  $F = dA$ , and Maxwell's equations read

$$0 = dF = d^*F.$$

(Here  $*$  is the Hodge duality operator; Hodge indeed introduced his celebrated theory of harmonic forms as a generalization to  $p$ -forms of Maxwell's equations). Yang-Mills theory or non-Abelian gauge theory can, at the classical level, be described similarly, with  $U(1)$  replaced by a more general compact gauge group  $G$ . The definition of the curvature must be modified to  $F = dA + A \wedge A$ , and Maxwell's equations are modified to the Yang-Mills equations,  $0 = d_A F = d_A * F$ , where  $d_A$  is the gauge covariant extension of the exterior derivative. These equations can be derived from the Yang-Mills Lagrangian

$$L = 1/4g^2 \int \text{Tr} F \wedge *F,$$

where  $\text{Tr}$  denotes an invariant quadratic form on the Lie algebra of  $G$ . The Yang-Mills equations are non-linear, so, in contrast to the Maxwell equations, but like the Einstein equations for the gravitational field, they are not explicitly soluble in general. However, they have certain properties in common with the Maxwell equations, and in particular, they

describe at the classical level massless waves that travel at the speed of the light.

Quantum Electrodynamics (QED) gave a very accurate account of quantum behavior of electromagnetic fields and forces. The question arose of whether the non-Abelian analog was important for describing other forces in nature, notably the weak force and the strong or nuclear force. The masslessness of classical Yang-Mills waves was a serious obstacle to applying Yang-Mills theory to the other forces, for the weak and nuclear forces are not associated with long-range fields or massless particles.

In the 1960's and 1970's, these obstacles to physical applications of non-Abelian gauge theory were overcome. In the case of the weak force, this was accomplished by the Weinberg-Salam-Glashow electroweak theory with gauge group  $H = SU(2) \times U(1)$ . The masslessness of classical Yang-Mills waves was avoided by elaborating the theory with an additional "Higgs field": this is a scalar field, transforming in a two-dimensional representation of  $H$ , whose non-zero and approximately constant value in the vacuum state reduces the structure group from  $H$  to a  $U(1)$  subgroup (diagonally embedded in  $SU(2) \times U(1)$ ). This theory describes both the electromagnetic and weak forces, in a more or less unified way; because of the reduction of the structure group to  $U(1)$ .

The solution of the problem of massless Yang-Mills fields for the strong interactions was of a completely different nature. The solution was not obtained by adding additional fields to Yang-Mills theory but by discovering a remarkable property of the quantum Yang-Mills theory itself (i.e., the quantum theory whose Lagrangian has been given above). This property is called "asymptotic freedom". When a quantum theory is asymptotically free, this means, roughly, that the quantum behavior at short distances is very similar to the classical behavior, but that the classical theory is not a good guide to the quantum behavior at long distances. Asymptotic freedom, together with other experimental and theoretical discoveries made in the 1960's and 70's made it possible to describe the nuclear force by a non-Abelian theory in which the group is  $G = SU(3)$ . The additional fields describe, at the classical level, "quarks", which are spin  $\frac{1}{2}$  objects somewhat analogous to the electron, but transforming in the fundamental representation of  $SU(3)$ . The non-Abelian gauge theory of the strong force is called Quantum Chromodynamics (QCD).

What should now be stressed is that classical non-Abelian gauge theory is very different from the observed world of strong interactions. For QCD to successfully describe the strong force, it must have at the quantum level the following three properties, each of which is dramatically different from the behavior of the classical theory: (i) It must have a "mass gap", that is, there must be some strictly positive constant  $\Delta$  such that every excitation of the vacuum has energy at least  $\Delta$ . This point is necessary to explain why the nuclear force is strong but short-ranged. (ii) It must have "quark confinement", that is, even though the theory is described in terms of elementary fields, such as the quarks, that transform non-trivially under  $SU(3)$ , the physical particle states – such as the proton, neutron, and spin – are  $SU(3)$ -invariant. This point is needed to explain why we never see individual quarks. (iii) It must have "chiral symmetry breaking", which means that the vacuum is potentially invariant under a certain subgroup of the full symmetry group that acts on the quark fields. This point is needed to account for the "current algebra" theory of the soft pions that was developed in the 1960's.

Ugo Moschella discussed and suggested a general framework for studying quantum field theory on the anti-de-Sitter (AdS) space-time, based on the assumption of positivity of the

spectrum of the possible energy operators. In this framework, he shown that the  $n$ -point functions are analytic in suitable domains of the complex AdS manifold, that it is possible to Wick rotate to the Euclidean manifold and come back, and that it is meaningful to restrict AdS quantum fields to Poincaré branes. He gave also a complete characterization, of two-point functions which are the simplest example of the theory. Moschella and co-workers proposed a plausible set of hypothesis for an interacting AdS QFT, among which the positivity of the spectrum of the energy operators, AdS-covariance and an adaptation of microcausality. The spectral hypothesis readily implies that the  $n$ -point correlation functions admit analytic continuations in tuboidal domains in the Cartesian product on  $n$  copies of (the covering of) the complexified AdS manifold. There are some analogies and discrepancies between the  $n$ -point tuboids of complex AdS and the corresponding ones of complex Minkowski space. One of the interesting points of the AdS geometry is that there exist families of submanifolds that can be identified with Minkowski space-times in one dimension less (branes): as a matter of fact, these submanifolds contain all the two-plane sections of parabolic-type of the AdS-quadric as the third class of timelike orbits. The Moschella and co-workers construction guarantees the possibility of considering restrictions of AdS quantum fields to these “Poincaré branes” and obtaining this way completely well-defined Minkowskian QFT’s. He stressed that this result is not as obvious as the well-known restrictibility of Minkowskian theories to lower dimensionality space-times because of the more complicated geometry. From a geometrical viewpoint, the conformal theories obtained under asymptotic scaling assumptions then also appear as limits of the previous Minkowskian QFT’s when the corresponding parabolic sections tend to infinity. This study also provides the opportunity of displaying some strange implications of the postulate of microcausality in the “pure AdS” case. In fact, the discrepancy between the pure AdS spacetime and the covering of AdS appears in a characteristic way in the classification of the two-point functions; it is revealed by the property of uniformity (or nonuniformity) of these functions in their analyticity domain  $\mathbb{C} \setminus [-1, +1]$  in the complex plane of the cosine of the AdS invariant distance. These phenomena introduce the more general problem of characterizing the interacting QFT’s on the pure AdS spacetime with respect to those on the covering.

– *The topology of three-dimensional space and the Perelman’s geometric ideas about the Ricci flow* (M. Boileau, M. Carfora).

Boileau talk and partly Carfora presentation focused on some relationships between topological theory of low-dimensional manifolds and the recent results by G. Perelman, containing in particular the complete proof of the *Poincaré’s Conjecture* (formulated by Henri Poincaré in 1904) and *Thurston’s Geometrization Conjecture* (1982). They particularly gave a clear exposition of those Perelman ideas and results that are intimately related to the Ricci flow program previously worked out by Richard Hamilton. More precisely, Perelman’s arguments rest on a foundation built by Hamilton with his study of the Ricci flow equation for Riemannian metrics. Indeed, Hamilton believed that Ricci flow could be used to establish the Poincaré Conjecture and more general topological classification results in dimension 3, and laid a program to accomplish this. The difficulty was to deal with singularities in the Ricci flow. Perelman’s breakthrough was to understand

the qualitative nature of the singularities sufficiently to allow him to prove the Poincaré Conjecture.

Recall that Poincaré conjecture states the following: *a closed, smooth, simply connected 3-manifold is diffeomorphic to  $S^3$* . Thurston Geometrization conjecture was stated as follows: *Let  $M$  be a closed, orientable, prime 3-manifold. Then the interior of any such manifold can be split in an essentially unique way by disjoint union of embedded 2-spheres, 2-tori and Klein bottles  $\coprod_i T_i^2 \subset M$  into pieces which have a geometric structure (other stated, such that every component of the complement admits a locally homogeneous Riemannian metric of finite volume)*. Here a “geometric structure” can be defined most easily as a complete Riemannian metric which is locally isometric to one of the eight model structures listed below.

Carfora shown how to introduce the conjugate linearized Ricci flow, a version of the linearization of the Ricci flow motivated by Perelman’s treatment of Ricci flow as a gradient flow. The flow takes into account the diffeomorphism invariance of the Ricci flow, which is the only obstruction to its strict parabolicity. He then characterized the conjugate linearized Ricci flow on closed three-manifolds of bounded geometry and discuss its properties. In particular, he expressed the evolution of the metric and of its Ricci tensor in terms of the backward heat kernel of the conjugate linearized Ricci flow. These results provide various conservation laws and monotonicity formulas for the linearized flow. It turns out that these results may be of interest to both analytical treatment of Ricci flow and physical applications of Ricci flow. In this last respect, Carfora mentioned the applications of the Ricci flow and Perelman’s entropy to the problem of averaging cosmological space-times, and to the characterization of the corresponding Friedmann-Lemaitre-Robertson-Walker templates.

M. Boileau covered many topics related with Perelman’s work and its connection with the fundamental question of how to classify and characterize hold and new three-dimensions manifold invariants. He showed how to construct the Ricci flow with surgery. He also discussed the long-time behaviour of the Ricci flow with surgery, which is needed for the full Geometrization conjecture. He showed, among other interesting things, that if the initial manifold is simply-connected then the Ricci flow with surgery becomes extinct in a finite time, thereby removing the issue of the long-time behavior. Combining this claim with the proof of the existence of Ricci flow with surgery gives the shortened proof in simply-connected case. Besides providing details for Perelman’s proofs, Boileau talk contained an overview of the Ricci flow approach to Geometrization of 3-manifolds.

One can employ Ricci flow in a more general study of three-manifolds than that usually used, which is based on the Thurston’s Geometrization Conjecture for three-manifolds stated above. Perelman has stated results which imply a positive resolution of Thurston’s Geometrization Conjecture. His proposed proof relies in an essentially way on the existence of Ricci flow with surgery for all positive time. But it also involves a further analysis of the limits of these Ricci flows as time goes to infinity. This further analysis involves analytic arguments, following earlier work of Hamilton in a simpler case of bounded curvature. They also involve a result from the theory of manifolds with curvature locally bounded below that are collapsed, related to previous results.

It is clear that Perelman’s argument did not arise in a vacuum. Firstly, it resides in a context provided by the general theory of Riemannian manifolds. In particular, various notions of convergence of sequences of manifolds play a crucial role. The most important is

geometric convergence (smooth convergence on compact subsets). Even more importantly, Perelman's argument resides in the context of the theory of the Ricci flow equation, introduced by Hamilton and extensively studied by him and others. Perelman makes use of almost every previously established result for 3-dimensional Ricci flows. One exception is Hamilton's proposed classification results for three-dimensional singularities. These are replaced by Perelman's strong qualitative description of singularity development for Ricci flow on compact three-manifolds.

The interesting talks by Claudio Bartocci and Joseph Kounieher essentially discussed the important question of the geometrical properties of integrable systems, and the role that these systems might have for elucidating some striking features of quantum field theories and particle physics, in particular the one related to the presence of physical dualities.

In his lecture, Sisir Roy has showed that the recent advances in the geometry at Planck scale clearly indicate that the kind of functional geometry he described is also a possible description for the geometry at small scale. Planck scale is considered to be the smallest scale in physical world depending on the values of fundamental constants like gravitational constant  $G$ , speed of light  $c$  and Planck constant  $h$ . The space, time or causality lose their meaning beyond Planck scale. The most challenging issue is how space-time at physical level (which is continuous) emerges from discrete structure. Here, the black holes occupying the Planck volume are supposed to be emitted or absorbed due to quantum fluctuation. So the functional states of black holes may generate the geometry at the physical level. The present authors along with Requardt Manfred described how macroscopic space-time or its underlying mesoscopic substratum emerges from a more fundamental concept, a fluctuating cellular network around the Planck scale. Here, the geometry emerges from a purely relational picture à la Leibniz. The discrete structure at the Planck scale consists of elementary nodes, which interact, or exchange information with each other via bonds playing the role of irreducible interactions. This model is a two-level system. The microscopic level, QX, is a dynamical cellular network of nodes and bonds. The macroscopic level, ST that self-organizes from QX is another cellular network, in which the nodes, or supernodes, are the cliques of a graph,  $G(t)$  of the QX level, bound in a network by superbonds.

Finally, Gabriel Catren and Alexei Grinbaum touched on different conceptual and philosophical aspects pertaining to the quantum world in general and to the loop quantum gravity theory in particular. Catren discussed the important issue of the nature of "physical reality" and the theoretical assumptions a complete description of the physical world should fulfil.

Let's conclude by stressing that the last three decades constituted a very fruitful time for theoretical physics. During this period the use of geometrical and topological methods and ideas have been particularly intense, leading to a new type of relation between physics and mathematics. Beginning in the eighties, we have witnessed how the most advanced developments in theoretical physics have led to new results in mathematics. A new type of relation emerged which is unprecedented in history. After a long period of divorce, which took place at the beginning of the 20th century when abstraction and rigour started to play a fundamental role in mathematics, the eighties constituted a period of reconciliation. (We should however mention two important exceptions, namely general relativity, which is based on Riemannian geometry, and the geometrical theory of dynamical systems worked out by Henri Poincaré and thereafter developed by several great mathematicians). Quantum field theory and string theory started to generate new mathematics, establishing a new type of relation. The most remarkable aspect of this new liaison is that the emerging new

mathematics were not produced because they were needed by physics but because they could lead to important breakthroughs in mathematics. Geometry and topology were the fields of mathematics that were particularly involved in these developments. On the physics side, the quantum aspects of field and string theory were the responsible for the establishment of the new connections in mathematics. All these developments have led to the study of new types of theories now known as topological field theories and topological string theories.